CAVITATION FLOW OF AN IDEAL INCOMPRESSIBLE FLUID IN A SLOT

(CAVITATSIONNOE TECHENIE IDEAL'NOI NESZHIMAEMOI ZHIDKOSTI V SHCHELI)

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We deal with a form of cavitation flow which is the general case of flow through a Borda orifice [1], and which can be used for a theoretical study of slot cavitation [2].

1. The general problem. Assume that a plane stream of ideal, weightless, incompressible fluid flows into an infinitely long slot, inside which there develops a cavitation void, i.e. a region of constant pressure bounded by a stream line which separates from the edge of the slot and goes off into a second sheet of a Riemann surface. Figure 1 is a diagram of this kind of cavitation flow. The letters V_1, V_2, and H denote, respectively, velocity of the approaching stream at infinity, velocity in the slot at infinity and the slot depth. The required flow has two critical points (location unknown at the outset), represented in Fig. 1 by the letters D and F. For the dimensions of the cavity we have: l the distance between two parallel tangents through points N and N_1 on the cavity and h the ordinate of the point M which has a tangent parallel to the upper wall.

The constant velocity at the edge or boundary of the cavity will be denoted by V_0 and the total flow of fluid through the slot is Q.

The flow we are dealing with is a general case, as mentioned above, of the well-known flow studied by Kirchoff [1]. It differs from the latter in that, instead of the classical Kirchoff-Holmholtz pattern [1], we choose that suggested in [3,4], and also, in our case, the velocity at infinity $V_1 \neq 0$.

Let us introduce two dimensionless geometrical parameters

$$\lambda = \frac{l}{H}, \qquad e = 1 - \frac{h}{H} \tag{1.1}$$

and we will call λ the relative length of the cavity and e the coefficient of contraction of the flow.

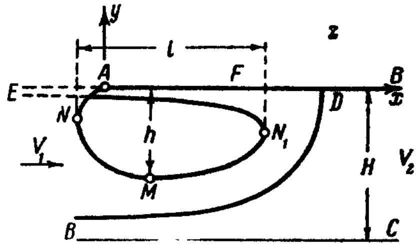

FIG. 1.

If we are given any two dimensionless physical quantities κ_1 and κ_2 which characterize the flow, then, in the physical sense of the problem, all remaining hydrodynamic flow parameters (in particular λ and e) should be determined by κ_1 and κ_2.

For κ_1 and κ_2 we can take the cavitation numbers

$$\varkappa_1 = \frac{p_1 - p_0}{{}^1/_2\rho V_1^2}, \qquad \varkappa_2 = \frac{p_2 - p_0}{{}^1/_2\rho V_2^2} \tag{1.2}$$

where p_0 is the pressure in the cavity, p_1 and p_2 pressures, respectively, at infinity upstream of the slot and within the slot.

Our problem involves the following: to find the complex potential of the required flow $W = W(z) = \phi + i\psi$, where ϕ is the velocity potential and ψ the stream function, and then to determine the relation between the quantities λ and e and the two dimensionless quantities κ_1 and κ_2.

2. Determination of complex potential. We will find the relation $W = W(z)$ in parametric form:

$$W = W(\zeta), \qquad z = z(\zeta) \tag{2.1}$$

where $\zeta = \xi + i\eta$ is a complex variable which covers the first quadrant of the upper half-plane $\xi\eta$. Put $W = 0$ at point A in the flow. The region in which W varies will be the half-plane with two cuts (Fig. 2). On transforming region W into region ζ so that the appropriate points in Figs. 2 and 3 correspond, we obtain (2.2)

$$W(\zeta) = \frac{Q}{\pi(d^2+f^2-\varepsilon^2)}\left[\zeta^2 - 1 - (d^2+f^2-\varepsilon^2)\ln\frac{\zeta^2-\varepsilon^2}{1-\varepsilon^2} + \frac{f^2d^2}{\varepsilon^2}\ln\frac{\zeta^2-\varepsilon^2}{(1-\varepsilon^2)\zeta^2}\right]$$

To determine $z(\zeta)$ we introduce an auxiliary function

$$\chi(\zeta) = \ln\frac{1}{V_0}\frac{dW}{dz} = \ln\frac{V}{V_0} - i\vartheta \tag{2.3}$$

and note that

$$z(\zeta) = \frac{Q}{\pi (d^2 + f^2 - \varepsilon^2)} \int\limits_1^\zeta \frac{(\zeta^2 - d^2)(\zeta^2 - f^2)}{\zeta(\zeta^2 - \varepsilon^2)} e^{-\chi(\zeta)} d\zeta \qquad (2.4)$$

It is easy to see that $\chi(\zeta)$ is an analytic function all over the region where ζ varies, with the exception of points $\zeta = f$ and $\zeta = d$, at which it has logarithmic singularities. At the boundaries of the regions where ζ varies, the function $\chi(\zeta)$ satisfies the following boundary conditions

$$\begin{array}{lll}
\operatorname{Im}\chi = 0 & \text{при} \begin{cases} \zeta = \xi, & 0 \leqslant \xi \leqslant d, \quad f \leqslant \xi \leqslant \infty \\ \zeta = i\eta, & 0 \leqslant \eta \leqslant \infty \end{cases} \\
\operatorname{Im}\chi = -\pi & \text{при } \zeta = \xi, & d \leqslant \xi \leqslant \varepsilon \\
\operatorname{Im}\chi = \pi & \text{при } \zeta = \xi, & 1 \leqslant \xi \leqslant f \\
\operatorname{Re}\chi = 0 & \text{при } \zeta = \xi, & \varepsilon \leqslant \xi \leqslant 1
\end{array} \qquad (2.5)$$

Thus, to determine the function $\chi(\zeta)$ it is necessary to solve the mixed boundary-value problem (2.5) which is easy to reduce to a mixed boundary-value problem in the upper half-plane.

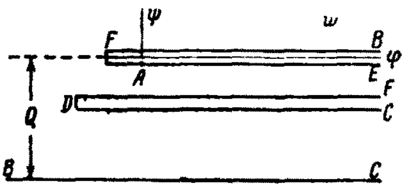

FIG. 2.

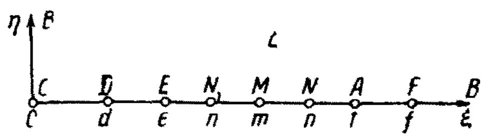

FIG. 3.

If we make use of formulas which yield a solution to the general boundary-value problem for a half-plane (5.6), and apply them to our case we obtain

$$\chi(\zeta) = \ln \frac{(1-\alpha^2)(1-\beta^2)(\zeta^2 - f^2)(\zeta^2 - d^2)}{(\sqrt{\zeta^2 - \varepsilon^2} + \beta\sqrt{\zeta^2 - 1})^2(\sqrt{\zeta^2 - 1} + \alpha\sqrt{\zeta^2 - \varepsilon^2})^2} \qquad (0 \leqslant \alpha,\ \beta \leqslant 1) \quad (2.6)$$

It is easy to see that the function $\chi(\zeta)$ which we have found satisfies all the conditions indicated above. The parameters α and β which enter expression (2.6), are connected with f and d by the relations

$$f^2 = \frac{1 - \alpha^2\varepsilon^2}{1 - \alpha^2}, \qquad d^2 = \frac{\varepsilon^2 - \beta^2}{1 - \beta^2} \qquad (2.7)$$

Instead of inserting $\chi(\zeta)$ into (2.4) we use (2.6) and find

$$z(\zeta) = \frac{2Hq}{\pi}\left[\frac{s}{2}(\zeta^2 - 1) - \frac{c}{2}\ln\frac{\zeta^2-\varepsilon^2}{1-\varepsilon^2} + \frac{b}{2\varepsilon^2}\ln\frac{1-\varepsilon^2/\zeta^2}{1-\varepsilon^2} + \right.$$

$$+ (\beta+\alpha)(1+\alpha\beta)\sqrt{(\zeta^2-1)(\zeta^2-\varepsilon^2)} - a\ln\frac{\sqrt{\zeta^2-1}+\sqrt{\zeta^2-\varepsilon^2}}{\sqrt{1-\varepsilon^2}} +$$

$$\left. + 2\,\frac{r}{\varepsilon}\ln\frac{\sqrt{\zeta^2-\varepsilon^2}+\varepsilon\sqrt{\zeta^2-1}}{\zeta\sqrt{1-\varepsilon^2}}\right] \tag{2.8}$$

where

$$a = (1+\alpha\beta)(\alpha(1+\varepsilon^2)+\beta(3-\varepsilon^2)), \quad b = (1+\alpha\beta)^2\varepsilon^2 + (\beta+\alpha\varepsilon)^2$$
$$c = (1+\alpha\beta)^2 + 2(\alpha+\beta)(\beta+\alpha\varepsilon^2) - (\alpha+\beta)^2\varepsilon^2, \quad r = (1+\alpha\beta)(\beta+\alpha\varepsilon^2)$$
$$q = 1:(a+c), \quad s = (1+\alpha\beta)^2 + (\alpha+\beta)^2 \tag{2.9}$$

3. Determination of parameters. With arbitrary values for the parameters α, β and ϵ which entered the formulas (2.2), (2.8) and (2.9), the solution obtained emerges as more general than was required at the outset. Indeed, it is easy to see that the formulas which give a solution to the problem would not change if line AFB on the one hand, and EDC on the other, represented solid parallel walls not touching each other (Fig. 4). It is therefore essential to require fulfilment of condition $\operatorname{Im} z(\zeta) = 0$ on EDC, or, as follows from (2.8), the condition

$$R(\alpha, \varepsilon)\beta^2 + \varepsilon T(\alpha, \varepsilon)\beta - \alpha\varepsilon^2 = 0$$

where

$$R(\alpha, \varepsilon) = (1+\varepsilon)^2 + (2+\varepsilon)\varepsilon\alpha$$
$$T(\alpha, \varepsilon) = 2 + (1-\alpha)\varepsilon$$

Solve this quadratic equation for β and we then have

$$\beta = \frac{\sqrt{T^2(\alpha,\varepsilon) + 4\alpha R(\alpha,\varepsilon)} - T(\alpha,\varepsilon)}{2T(\alpha,\varepsilon)} \tag{3.1}$$

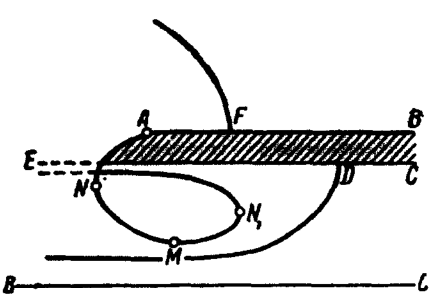

FIG. 4.

Thus, the solution we have found contains the two arbitrary parameters α and ϵ. To determine these parameters one must be given the two aforementioned physical quantities κ_1 and κ_2. If by using the complex potential $W = W(z)$ found in parametric form (2.1) it is possible to express κ_1 and κ_2 in terms of a and ϵ then we will have two equations for determining a and ϵ :

$$\varkappa_1 = \varkappa_1(\alpha, \varepsilon;\ \beta(\alpha, \varepsilon)), \qquad \varkappa_2 = \varkappa_2(\alpha, \varepsilon;\ \beta(\alpha, \varepsilon))$$

where $\beta(a, \epsilon)$ is the R.H.S. of Equation (3.1).

Let κ_1 and κ_2 be cavitation numbers determined by formulas (1.2); we then have

$$\varkappa_1 = \frac{(1+\alpha)^2\,(1+\beta(\alpha,\varepsilon))^2}{(1-\alpha)^2\,(1-\beta(\alpha,\varepsilon))^2} - 1, \qquad \varkappa_2 = \frac{(1+\alpha\varepsilon)^2\,(\varepsilon+\beta(\alpha,\varepsilon))^2}{(1-\alpha\varepsilon)^2\,(\varepsilon-\beta(\alpha,\varepsilon))^2} - 1 \tag{3.2}$$

We are convinced of the validity of these equations if we recognise, in virtue of the Bernoulli integral

$$p_1 - p_0 = \frac{1}{2}\rho\,(V_0^2 - V_1^2), \qquad p_2 - p_0 = \frac{1}{2}\rho\,(V_0^2 - V_2^2)$$

and that, in accordance with (2.3)

$$V_1 / V_0 = \exp \chi(\infty), \qquad V_2 / V_0 = \exp \chi(0)$$

4. Shape and dimensions of the cavity contraction coefficient. The shape of the cavity is represented by the free stream $ANMM_1E$ which bounds it (Fig. 1).

In order to rind the shape of the free stream line $X = X(\xi)$, $Y = Y(\xi)$ let us put $\zeta = \xi$, $\epsilon \leqslant \xi \leqslant 1$ in Equation (2.8) and separate the real and imaginary parts; we then get

$$X(\xi) = \frac{Hq}{\pi}\left[s(\xi^2-1) - c\ln\frac{\xi^2-\varepsilon^2}{1-\varepsilon^2} + \frac{b}{\varepsilon^2}\ln\frac{\xi^2-\varepsilon^2}{(1-\varepsilon^2)\,\xi^2}\right] \quad (\varepsilon \leqslant \xi \leqslant 1)$$

$$Y(\xi) = \frac{2Hq}{\pi}\Bigg[(\beta+\alpha)(1+\alpha\beta)\sqrt{(1-\xi^2)(\xi^2-\varepsilon^2)} - - a \operatorname{arc\,tg}\sqrt{\frac{1-\xi^2}{\xi^2-\varepsilon^2}} + 2\frac{r}{\varepsilon}\operatorname{arc\,tg}\varepsilon\sqrt{\frac{1-\xi^2}{\xi^2-\varepsilon^2}}\Bigg] \tag{4.1}$$

It is evident that the free stream has two tangents parallel to the y axis at points $N(X(n), Y(n))$, $N_1(X(n_1), Y(n_1))$ and one tangent parallel to the x axis at $M(X(m), Y(m))$, so that

$$n^2,\ n_1^2 = \frac{c+\varepsilon^2 s \pm \sqrt{(c+\varepsilon^2 s)^2 - 4sb}}{2s}, \qquad m^2 = \frac{\beta + \alpha\varepsilon^2}{\alpha+\beta} \tag{4.2}$$

In Equation (1.1), if we assume $l = X(n_1) - X(n)$, $h = -Y(m)$, then using formulas (4.1) and (4.2) we find the relative length of the cavity

$$\lambda = \frac{q}{\pi}\left[\frac{b}{\varepsilon^2}\ln\frac{n^2}{n_1^2} + \left(c - \frac{b}{\varepsilon^2}\right)\ln\frac{n^2-\varepsilon^2}{n_1^2-\varepsilon^2} - s\,(n^2 - n_1^2)\right] \tag{4.3}$$

and the coefficient of contraction is

$$e = 1 - \frac{2g}{\pi}\left[\, a \operatorname{arc\,tg}\sqrt{\frac{1-m^2}{m^2-\varepsilon^2}} - 2\frac{r}{\varepsilon}\operatorname{arc\,tg}\varepsilon\sqrt{\frac{1-m^2}{m^2-\varepsilon^2}} - \right.$$
$$\left. - (\beta+\alpha)(1+\alpha\beta)\sqrt{(1-m^2)(m^2-\varepsilon^2)}\,\right] \tag{4.4}$$

The relative depth of the cavity h/H is found from (1.1).

5. Particular cases. (a) If we put $a = 1$ in formulas (2.7)-(4.4) we obtain a solution of the problem of flow of fluid, at rest at infinity, through an infinitely long slot with the formation of a finite cavity.

In this case we will obtain

$$z(\zeta) = \frac{H}{\pi}\left[\,\sigma(\zeta^2-1) - \frac{1}{2}\ln\frac{\zeta^2-\varepsilon^2}{1-\varepsilon^2} + (1-\tau)\ln\frac{\zeta^2-\varepsilon^2}{(1-\varepsilon^2)\zeta^2} + \right. \tag{5.1}$$
$$\left. + \sigma\sqrt{(\zeta^2-1)(\zeta^2-\varepsilon^2)} - \ln\frac{\sqrt{\zeta^2-1}+\sqrt{\zeta^2-\varepsilon^2}}{\sqrt{1-\varepsilon^2}} + 2\tau\ln\frac{\sqrt{\zeta^2-\varepsilon^2}+\varepsilon\sqrt{\zeta^2-1}}{\zeta\sqrt{1-\varepsilon^2}}\right]$$

$$W(\zeta) = -\frac{Qd^2}{\pi\varepsilon^2}\left[\left(\frac{\varepsilon^2}{d^2}-1\right)\ln\frac{\zeta^2-\varepsilon^2}{1-\varepsilon^2} + 2\ln\zeta\,\right] \tag{5.2}$$

$$\lambda = \frac{1}{\pi}\left[(1-\tau)\ln\frac{n^2}{n_1^2} + \left(\tau - \frac{1}{2}\right)\ln\frac{n^2-\varepsilon^2}{n_1^2-\varepsilon^2} - \sigma(n^2 - n_1^2)\right] \tag{5.3}$$

$$e = 1 - \frac{1}{\pi}\left(\operatorname{arc\,tg}\sqrt{\frac{\omega}{\varepsilon}} + 2\tau\operatorname{arc\,tg}\sqrt{\varepsilon\omega} - \frac{1+\varepsilon}{1+\sqrt{2}}\sqrt{\varepsilon\omega}\right) \tag{5.4}$$

where

$$\sigma = 1 : (1 + (\sqrt{2}-1)\varepsilon)^2, \qquad \tau = (1+\varepsilon\sqrt{2}) : (1 + (\sqrt{2}-1)\varepsilon)^2$$
$$\omega = 1 + \sqrt{2} + \sqrt{2}\varepsilon, \qquad d^2 = 2\varepsilon(1 + (\sqrt{2}-1)\varepsilon) : (\sqrt{2}+1)(1 + (\sqrt{2}-1)^2\varepsilon) \tag{5.5}$$

$$n^2,\ n_1^2 = \frac{2\sigma\varepsilon^2 + 1 \pm \sqrt{(2\sigma\varepsilon^2+1)^2 - 16(1-\tau)\varepsilon^2}}{4\sigma} \tag{5.6}$$

It can easily be seen that in this flow the velocity at infinity before the slot V vanishes and the critical point F is removed to infinity ($f = \infty$); because only one parameter ϵ enters into the solution and, in order to find it, we have to be given one dimensionless physical quantity $\kappa = \kappa(\epsilon)$.

If κ is the cavitation number $\kappa = (p_2 - p_0)/0.5\rho V_2^2$ we have

$$\varkappa = \frac{2(\sqrt{2}+1)(1+\sqrt{2}\varepsilon)}{(1-\varepsilon)^2}, \qquad \varepsilon = \frac{\sqrt{\varkappa-1} - (\sqrt{2}+1)}{1+\sqrt{\varkappa-1}}$$

(b) We can obtain another particular case if, in formulas (2.2)-(2.9), (3.2)-(4.4) we put $\beta = 0$, $\epsilon = 0$. Then we will have

$$z(\zeta) = \frac{H}{\pi(1+\alpha)}\left[(1+\alpha^2)(\zeta^2-1) - 2\ln\zeta + 2\alpha\left(\zeta\sqrt{\zeta^2-1} + \ln\frac{1}{\zeta+\sqrt{\zeta^2-1}}\right)\right] \tag{5.7}$$

$$w(\zeta) = \frac{Q}{\pi}\left[(1-\alpha^2)(\zeta^2-1) - 2\ln\zeta\right] \tag{5.8}$$

$$\lambda = \infty, \qquad e = 1 : (1+\alpha) \tag{5.9}$$

$$m = 0, \qquad n^2 = 1 : (1+\alpha), \qquad n_1 = 0 \tag{5.10}$$

In this flow the velocity in front of the slot at infinity is not zero and there is a critical point F on the wall. Thus, formulas (5.7)-(5.10) yield the solution to the problem of a fluid moving at infinity into an infinitely long slot (Fig. 5). Into the solution there enters one parameter a and, therefore, to determine this parameter we must be

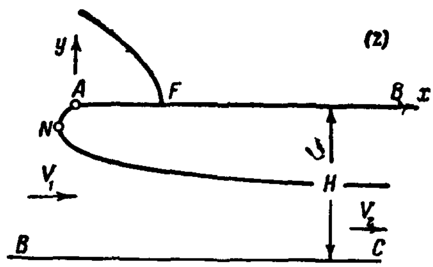

FIG. 5.

given one dimensionless physical quantity $\kappa = \kappa(a)$. If κ is the cavitation number, $\kappa = (p_1 - p_0)/0.5\rho V_1^2$ so that

$$\varkappa = \frac{4\alpha}{(1-\alpha)^2}, \qquad \alpha = \frac{\varkappa}{2+\varkappa+2\sqrt{1+\varkappa}}$$

(c) Assuming $\epsilon = 0$ in formulas (5.1)-(5.6), or $a = 1$ in formulas (5.7)-(5.10), we obtain

$$z(\zeta) = \frac{H}{\pi}\left(\zeta^2 - 1 - \ln\zeta + \zeta\sqrt{\zeta^2-1} + \ln\frac{1}{\zeta+\sqrt{\zeta^2-1}}\right)$$

$$w(\zeta) = -\frac{2Q}{\pi}\ln\zeta, \qquad e = \frac{1}{2}$$

i.e. the solution to the well-known problem of the flow of fluid, at rest at infinity, into an infinitely long slot [1].

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